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The Ising–Sherrington-Kirpatrick Model in a Magnetic Field at High Temperature

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We study a spin system on a large box with both Ising interaction and Sherrington-Kirpatrick couplings, in the presence of an external field. Our results are: (i) existence of the pressure in the limit of an infinite box. When both Ising and Sherrington-Kirpatrick temperatures are high enough, we prove that: (ii) the value of the pressure is given by a suitable replica symmetric solution, and (iii) the fluctuations of the pressure are of order of the inverse of the square of the volume with a normal distribution in the limit. In this regime, the pressure can be expressed in terms of random field Ising models.

KEY WORDS: Ising model; Sherrington-Kirpatrick model; spin-glass; thermodynamic limit; pressure; quadratic coupling.

1. INTRODUCTION

We consider a d-dimensional Ising model in a magnetic field h, perturbed by a mean field interaction of spin-glass type. The Hamiltonian contains two parameters, β and κ, which play the role of two inverse temperatures. When $\beta = 0$ the model reduces to the Ising model at temperature $1/\kappa$, while for $\kappa = 0$ one recovers the Sherrington-Kirkpatrick (SK) model at temperature $1/\beta$. The understanding of the SK model has recently witnessed great progress (see, e.g., refs. 8, 10 and 14). The main interest in the analysis of this model is the possibility of investigating the robustness

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of the phenomena typical of mean field spin glass models, in the presence of additional interactions of non-mean-field character. In addition, the "Ising–Sherrington-Kirpatrick" model is possibly of physical interest in itself since, as well known, the oscillating part of the interaction in real spin glasses decays quite slowly with distance.

The model has been previously considered in ref. 3 under the additional assumptions that $h=0$ and that the Ising model is ferromagnetic. Under these conditions it was proven that, if β and κ are small enough, the infinite volume pressure is given by the sum of the Ising pressure and of the SK one, at the respective temperatures. Moreover, the disorder fluctuations of the pressure were found to be of order $1/V$, V being the volume of the system, and to satisfy a central limit theorem. (Actually, the fluctuations are still of order $1/V$ even if the ferromagnetic requirement is dropped, while the proof of the central limit behavior in ref. 3 requires the validity of the FKG inequalities).

In the present paper, the Ising interaction decays exponentially fast with distance, but is not necessarily ferromagnetic. It turns out that the presence of the magnetic field changes qualitatively the picture with respect to ref. 3. Indeed we find, still for κ and β small enough, that the limit pressure is given in terms of the pressure of an Ising model with random external field, the strength of the randomness being related to the typical value of the overlap between two replicas of the system. It is remarkable, though natural, that the random field Ising model, which has its own interest,[2] plays such an important role in our model. Note also that the pressure of our model can be computed only via the thermodynamic limit of another disordered system. This is contrast with the case $h=0$ we mentioned above, and also of course with the case $\kappa=0$ of the standard SK model. The second difference is that the fluctuations of the pressure in presence of h satisfy a central limit theorem on the scale $1/\sqrt{V}$ rather than $1/V$. The same phenomenon is known to happen in the case of the usual SK model, (see for instance refs. 1, 4 and 11).

In the region of thermodynamic parameters we consider, the system is in a "replica-symmetric" (RS) phase, the overlap between two independent replicas being a non-random value in the thermodynamic limit. Our methods fail beyond some values $\beta_0(h)$ and $\kappa_0(h)$, which we believe to be an artifact of our approach, rather than representing the true boundary of the RS region. The same inconvenient has been previously encountered in the analysis of the SK model.[9,15] In principle one could improve these values by employing the replica symmetry breaking scheme which for instance enabled M. Talagrand[16] to control the whole RS region of the SK model, and later the entire phase space.[14] However, in the present case (as well as in ref. 3) one of the reasons why we do not reach the true critical line is due to an incomplete control of the underlying random field Ising model, and this problem would not be fixed by the methods of ref. 16. For this reason, we prefer to use a generalization of the technically simpler "quadratic replica coupling" technique introduced in ref. 9.

It would be of course an interesting challenge to go beyond the present approach, and to deal with lower-temperature situations, where the Ising–SK system possibly shows a RSB-like behavior.

2. DESCRIPTION OF THE MODEL AND RESULTS

The model we consider is defined on the d-dimensional hypercubic box $\Lambda_N = \{-N, \ldots, N\}^d$ and its partition function is:

$$Z_N(\kappa, \beta, h; J) = \sum_{\sigma \in \{-1,+1\}^{\Lambda_N}} \exp\left(-\kappa H_N^I(\sigma) - \beta H_N^{SK}(\sigma; J) + h \sum_{i \in \Lambda_N} \sigma_i\right). \tag{1}$$

The Hamiltonian of the SK model is defined as

$$H_N^{SK}(\sigma; J) = -\frac{1}{\sqrt{2|\Lambda_N|}} \sum_{i,j \in \Lambda_N} J_{ij} \sigma_i \sigma_j$$

and the couplings J_{ij} are i.i.d. Gaussian random variables $\mathcal{N}(0, 1)$. On the other hand, the Hamiltonian of the Ising model is

$$H_N^I(\sigma) = -\frac{1}{2} \sum_{i,j \in \Lambda_N} K(i-j) \sigma_i \sigma_j,$$

where we assume that the interaction decays exponentially, i.e.,

$$|K(i)| \leqslant C_1 e^{-C_2 |i|} \tag{2}$$

for some $C_1, C_2 > 0$. We do not require the interaction to be ferromagnetic. The finite volume (disorder-dependent) pressure is defined as usual as

$$p_N(\kappa, \beta, h; J) = \frac{1}{|\Lambda_N|} \log Z_N(\kappa, \beta, h; J).$$

Later, we will need to consider the (Gaussian) Random Field Ising Model (RFIM), defined by the partition function

$$Z_N^{\mathrm{RFIM}}(\kappa, h, \gamma; J) = \sum_{\sigma \in \{-1,+1\}^{\Lambda_N}} \exp\left(-\kappa H_N^I(\sigma) + \sum_{i \in \Lambda_N} \sigma_i (h + \gamma J_i)\right),$$

J_i being i.i.d. standard Gaussian variables, and being also independent from the J_{ij}'s in the sequel. The existence of the infinite volume pressure of the RFIM,

$$\begin{aligned} p^{\mathrm{RFIM}}(\kappa, h, \gamma) &= \lim_{N \to \infty} \mathbb{E}\, p_N^{\mathrm{RFIM}}(\kappa, h, \gamma; J) \\ &= a.s. - \lim_{N \to \infty} p_N^{\mathrm{RFIM}}(\kappa, h, \gamma; J), \end{aligned}$$

is a well known consequence of additivity and of the ergodic theorem, e.g. ref. 17.

Our main results can be summarized as follows:

Theorem 1. For all $h \in \mathbb{R}$, and all $\kappa, \beta > 0$, the limit

$$p(\kappa, \beta, h) = \lim_{N \to \infty} \mathbb{E}\, p_N(\kappa, \beta, h; J), \tag{3}$$

exists and

$$p_N(\kappa, \beta, h; J) \overset{N \to \infty}{\longrightarrow} p(\kappa, \beta, h) \tag{4}$$

for a.e. J and in the L^p-norm ($p \in [1, \infty)$).

Theorem 2. For all $h \in \mathbb{R}$, there exist $\kappa_0(h) > 0$ and $\beta_0(h) > 0$ such that, for $0 \leqslant \kappa \leqslant \kappa_0(h)$ and $0 \leqslant \beta \leqslant \beta_0(h)$

$$p(\kappa, \beta, h) = \inf_{0 \leqslant q \leqslant 1} \left(p^{\mathrm{RFIM}}(\kappa, h, \beta\sqrt{q}) + \frac{\beta^2}{4}(1-q)^2\right). \tag{5}$$

In the case $\kappa = 0$, it is established in ref. 7 that the infimum in (5) is achieved at a unique point. As explained below (see Section 3.2), we take small enough $\kappa_0(h)$ so that, for $0 \leqslant \kappa \leqslant \kappa_0(h)$ and $0 \leqslant \beta \leqslant \beta_0(h)$, the infimum in (5) is achieved at a unique q. In the following, we will always denote by $\bar{q} = \bar{q}(\kappa, \beta, h)$ the value that realizes the infimum in (5).

We emphasize that $\kappa_0(h)$ is taken small enough so that the RFIM is inside Dobrushin's uniqueness region for every realization of the external fields $J_i, i \in \mathbb{Z}^d$. Let $\langle\cdot\rangle_{\infty,J}$ be the unique infinite volume Gibbs measure for the RFIM with $\gamma=\beta\sqrt{\bar{q}}$, depending on $J_i, i\in\mathbb{Z}^d$. If $\preceq$ denotes the lexicographic order in $\mathbb{Z}^d$, define

$$\Gamma=\Gamma(\kappa,\beta,h)=\mathbb{E}_{J_i,i\succeq 0}\left(\mathbb{E}_{J_0',J_i,i\prec 0}\log\left\langle e^{\beta\sqrt{\bar{q}}(J_0'-J_0)\sigma_0}\right\rangle_{\infty,J}\right)^2,$$

where J_0' is an independent copy of J_0, J_0' being independent of $(J_i, i\in\mathbb{Z}^d)$. Here, the $\mathbb{E}$-expectations are conditional, and the subscripts of $\mathbb{E}$ indicate on which variables the expectation is performed. Then, we have the following central limit theorem:

Theorem 3. For $0\leqslant\kappa\leqslant\kappa_0(h)$ and $0\leqslant\beta\leqslant\beta_0(h)$,

$$\sqrt{|\Lambda_N|}\Big(p_N(\kappa,\beta,h;J)-\mathbb{E}\,p_N(\kappa,\beta,h;J)\Big)\xrightarrow{\text{law}}\mathcal{N}\left(0,\Gamma-\frac{\beta^2}{2}\bar{q}^2\right). \qquad (6)$$

One can check by expansion around $\kappa=\beta=0, h\neq 0$, that the limiting variance $\Gamma-(\beta^2/2)\bar{q}^2$ is strictly positive for any fixed non-zero h and small κ,β. This proves that the fluctuations of $p_N(\kappa,\beta,h;J)$ are truly of order of the square root of the volume inverse in this region of the parameters. To match with the breakdown in the order of magnitude of fluctuations at zero external field, we observe that both $\bar{q}$ and Γ vanish as $h\to 0$, and also that they are equal to zero when $h=0$.

3. PROOFS

3.1. For Small κ the RFIM is Inside the Dobrushin Uniqueness Region

Let $i\neq k$ be lattice points. Under any (infinite volume) RFIM Gibbs measure $\langle\cdot\rangle_{\infty,J}$ the law $\langle\cdot|\eta\rangle_{i,\infty,J}$ of σ_i given $\sigma_j=\eta_j, j\neq i$, is Bernoulli with parameter proportional to $\exp\{\sigma_i[\mathcal{H}_{i,k;\eta}+\kappa K(i-k)\eta_k]\}$ with

$$\mathcal{H}_{i,k;\sigma}=\kappa\sum_{j\neq i,k}K(i-j)\sigma_j+h+\gamma J_i.$$

Following e.g. Section 2 in ref. 5, we define the Dobrushin's influence coefficient

$$C_{ki}=\sup\left\{\frac{1}{2}\|\,\langle\cdot|\eta\rangle_{i,\infty,J}-\left\langle\cdot|\eta'\right\rangle_{i,\infty,J}\|_{\text{var}};\,\eta=\eta'\text{ off }k\right\},$$

where $\|\cdot\|_{\rm var}$ is the variation norm. With a straightforward computation,

$$\begin{aligned}\| \langle\cdot|\eta\rangle_{i,\infty,J} - \langle\cdot|\eta'\rangle_{i,\infty,J} \|_{\rm var} &= 2|\,\langle+|\eta\rangle_{i,\infty,J} - \langle+|\eta'\rangle_{i,\infty,J}\,| \\ &\leqslant \frac{2|\sinh 2\kappa K(i-k)|}{\cosh[2\mathcal{H}_{i,k;\eta}]+\cosh[2\kappa K(i-k)]}\end{aligned}$$

for such η, η', and so

$$C_{ki} \leqslant |\tanh 2\kappa K(i-k)| \leqslant |2\kappa K(i-k)|. \tag{7}$$

Therefore, for $\kappa < \kappa_1 = (2\sum_{i\neq 0}|K(i)|)^{-1}$ we derive from (7) and (2) that $a = \sup_i \sum_k \rho^{|i-k|} C_{ki} < 1$ for some $\rho > 1$, which implies (see Section 2.3 in ref. 5) that the Gibbs measure $\langle\cdot\rangle_{\infty,J}$ is unique and has exponentially decreasing correlations. More precisely, for all local functions f, g, there is a finite constant $C = C(a, f, g)$ such that for all $i \in \mathbb{Z}^d$

$$|\,\langle f; g\circ\theta_i\rangle_{\infty,J}\,| \leqslant C\rho^{-|i|} \tag{8}$$

with θ_i the shift of vector i and $\langle f; g\rangle_{\infty,J}$ the covariance of f, g.

3.2. Uniqueness of $\bar{q}$

Introduce

$$F(q,\kappa) = F_{\beta,h}(q,\kappa) = p^{\rm RFIM}(\kappa, h, \beta\sqrt{q}) + \frac{\beta^2}{4}(1-q)^2,$$

which is, in view of (8), a smooth function of all its arguments if $\kappa < \kappa_1$ (e.g., Corollary 8.37 in ref. 6).

Proposition 1. Assume $h \neq 0$. For all β, there exists $\kappa_2 = \kappa_2(\beta, h) > 0$ such that $q \mapsto F(q,\kappa)$ has a unique minimizer on [0, 1] for $\kappa < \kappa_2$.

By Guerra,[(7)] we know that the minimizer q_0 of $F(q,0)$ is unique, strictly positive and that[4]

$$\frac{\partial^2}{\partial^2 q}F(q_0, 0) > 0. \tag{9}$$

[4]Ref. 7, p. 166.

By (8), the function F is continuous in κ uniformly in $q\in(0,1]$. Hence,

$$\forall\delta>0\ \exists\tilde{\kappa}:\forall\kappa<\tilde{\kappa}\quad \arg\min_{[0,1]}F(\cdot,\kappa)\subset(q_0-\delta,q_0+\delta). \tag{10}$$

Again by (8), the function F is C^2 in a neighborhood of $(q_0,0)$. By the implicit function theorem for the equation

$$\frac{\partial}{\partial q}F(q,\kappa)=0,$$

(which applies thanks to condition (9)) we then derive that there exist neighborhoods U of q_0 and V of $\kappa=0$ and a function $\bar{q}:V\mapsto U$ such that, for $q\in V,\kappa\in U$, the above equation is equivalent to $q=\bar{q}(\kappa)$. With δ small enough so that $(q_0-\delta,q_0+\delta)\subset V$, we choose now $\kappa_2<\tilde{\kappa}$ (with $\tilde{\kappa}$ from (10)) such that $(-\kappa_2,\kappa_2)\subset U$. Then, for $\kappa<\kappa_2$, $F(\cdot,\kappa)$ has a unique minimum at $q=\bar{q}(\kappa)$.

As for the case $h=0$, one can prove similarly the following results, that we mention for comparison but will neither prove nor use.

Proposition 2. For all $\beta\neq 1$, there exists $\kappa_3=\kappa_3(\beta)>0$ such that $q\mapsto F_{\beta,0}(q,\kappa)$ has a unique minimizer on [0, 1] for $\kappa<\kappa_3$. If $\beta<1$, the minimizer is $q=0$.

Remark. The restriction $\beta\neq 1$ is due to the fact that for $\beta=\beta_c=1$ (the critical point of the SK model) one has $q_0(\beta_c,0)=0$ and, in contrast with (9),

$$\frac{\partial^2}{\partial^2 q}F_{\beta_c,0}(0,0)=0.$$

3.3. Proof of Theorem 1

Proof of (3). The proof is standard, we just sketch the main steps. Consider the box Λ_{mN}, with $m,N\in\mathbb{N}$, and partition it into sub-boxes $\Lambda_N^{(\ell)}$, $\ell=1,\dots,m^d$, congruent to Λ_N. Moreover, let $\tilde{Z}_{mN}$ be the partition function of the system where, with respect to (1), $H_{mN}^{SK}(\sigma;J)$ is replaced by

$$-\sum_{\ell=1}^{m^d}\frac{1}{\sqrt{2|\Lambda_N|}}\sum_{i,j\in\Lambda_N^{(\ell)}}J_{ij}^{(\ell)}\sigma_i\sigma_j$$

the $J_{ij}^{(\ell)}$ being m^d independent families of standard Gaussian variables. Note that, in this system, the different sub-boxes interact only through the Ising potential $K(.)$. Then, following the ideas of ref. 10, it is easy to prove that

$$p_{mN}(\kappa, \beta, h) \geqslant \frac{1}{|\Lambda_{mN}|} \mathbb{E} \log \tilde{Z}_{mN}(\kappa, \beta, h; J^{(\ell)}).$$

Since the Ising potential is summable, the interaction among the different sub-boxes due to the potential K grows at most proportionally to κ and to the the total surface, $d\, m^d N^{d-1}$. As a consequence, one has the approximate monotonicity

$$p_{mN}(\kappa, \beta, h) \geqslant p_N(\kappa, \beta, h) - \kappa \frac{C}{N}$$

for some constant C depending on the potential $K(.)$. From this, it is a standard fact to deduce that the sequence $\{p_{mN}\}_N$ has a limit when $N \to \infty$, that it does not depend on m, and that it coincides with $\lim_N p_N$.

Proof of (4). The almost sure convergence is standard and follows from exponential self-averaging of the pressure (see for instance Proposition 2.18 of ref. 12), which in the present case reads

$$\mathbb{P}\left(|p_N(\kappa, \beta, h; J) - \mathbb{E}\, p_N(\kappa, \beta, h; J)| \geqslant u\right) \leqslant D_1\, e^{-D_2(\beta)|\Lambda_N| u^2},$$

together with Borel–Cantelli's lemma. The L^p-convergence comes from uniform integrability, which again follows from exponential concentration.

3.4. Proof of Theorem 2

For $0 \leqslant t \leqslant 1$ and any $q \geqslant 0$, define the interpolating partition function

$$Z(t) = \sum_{\sigma \in \{-1,+1\}^{\Lambda_N}} \exp\left(-H^{(t)}(\sigma)\right), \tag{11}$$

$$H^{(t)}(\sigma) = \kappa H_N^I(\sigma) + \beta\sqrt{t} H_N^{SK}(\sigma; J) - \sum_{i \in \Lambda_N} \sigma_i \left(h + \beta\sqrt{q(1-t)} J_i\right)$$

with the properties

$$Z(0) = Z_N^{\mathrm{RFIM}}(\kappa, h, \beta\sqrt{q}; J),$$
$$Z(1) = Z_N(\kappa, \beta, h; J).$$

The t-derivative of the corresponding pressure

$$p_N(t) = \frac{1}{|\Lambda_N|}\mathbb{E}\log Z(t)$$

is easily computed: We denote by $\langle\cdot\rangle_t$ the Gibbs measure associated to $H^{(t)}$, by $\langle\cdot\rangle_t^{\otimes 2}$ its tensor product acting on a pair $(\sigma^1, \sigma^2) \in \{-1, +1\}^{\Lambda_N} \times \{-1, +1\}^{\Lambda_N}$, by $q_{12} = |\Lambda_N|^{-1}\sum_{i\in\Lambda_N}\sigma_i^1\sigma_i^2$ the overlap between configurations σ^1 and σ^2, and we get by the Gaussian integration by parts formula $\mathbb{E}\, J_i F(J_i) = \mathbb{E}\, F'(J_i)$,

$$\begin{aligned}\frac{d}{dt}p_N(t) &= \frac{\beta}{2|\Lambda_N|}\mathbb{E}\left\{\frac{1}{\sqrt{2|\Lambda_N|t}}\sum_{i,j\in\Lambda_N} J_{ij}\left\langle\sigma_i\sigma_j\right\rangle_t - \frac{\sqrt{q}}{\sqrt{1-t}}\sum_{i\in\Lambda_N} J_i\left\langle\sigma_i\right\rangle_t\right\}\\ &\overset{\text{int.by parts}}{=} \frac{\beta}{2|\Lambda_N|}\mathbb{E}\left\{\frac{\beta}{2|\Lambda_N|}\sum_{i,j\in\Lambda_N}\left(1-\left\langle\sigma_i^1\sigma_j^1\sigma_i^2\sigma_j^2\right\rangle_t^{\otimes 2}\right)\right.\\ &\quad\left. -\beta q\sum_{i\in\Lambda_N}\left(1-\left\langle\sigma_i^1\sigma_i^2\right\rangle_t^{\otimes 2}\right)\right\} = \frac{\beta^2}{4}(1-q)^2 - \frac{\beta^2}{4}\mathbb{E}\left\langle(q_{12}-q)^2\right\rangle_t^{\otimes 2},\end{aligned} \tag{12}$$

so that, integrating in t between 0 and 1, taking the $N \to \infty$ limit and optimizing on q, we have the "first half" of Eq. (5):

$$p(\kappa, \beta, h) \leqslant \inf_{q\geqslant 0}\left(p^{\mathrm{RFIM}}(\kappa, h, \beta\sqrt{q}) + \frac{\beta^2}{4}(1-q)^2\right). \tag{13}$$

Note that the inequality holds for any values of β and κ.

Next, consider a system of two coupled replicas: For $\lambda > 0$ let

$$Z^{(2)}(t, \lambda) = \sum_{\sigma^1, \sigma^2 \in \{-1,+1\}^{\Lambda_N}} \exp\left(-H^{(t)}(\sigma^1) - H^{(t)}(\sigma^2) + \frac{\beta^2}{2}|\Lambda_N|\lambda(q_{12}-q)^2\right)$$

and denote by $\langle\cdot\rangle_{t,\lambda}$ the Gibbs measure on $\{-1,+1\}^{\Lambda_N}\times\{-1,+1\}^{\Lambda_N}$ associated to this partition function. (Of course, $\langle\cdot\rangle_{t,0}=\langle\cdot\rangle_t^{\otimes 2}$ and $Z^{(2)}(t,0)=Z(t)^2$.) With

$$p_N^{(2)}(t,\lambda)=\frac{1}{2|\Lambda_N|}\mathbb{E}\log Z^{(2)}(t,\lambda),$$

we have by a computation similar to (12) and using symmetry in σ^1,σ^2,

$$\begin{aligned}
&\frac{d}{dt}p_N^{(2)}(t,\lambda_0-t)\\
&=\frac{\beta}{4|\Lambda_N|}\mathbb{E}\left\{\frac{2}{\sqrt{2|\Lambda_N|t}}\sum_{i,j\in\Lambda_N}J_{ij}\left\langle\sigma_i^1\sigma_j^1\right\rangle_{t,\lambda_0-t}\right.\\
&\qquad\left.-\frac{2\sqrt{q}}{\sqrt{1-t}}\sum_{i\in\Lambda_N}J_i\left\langle\sigma_i^1\right\rangle_{t,\lambda_0-t}-\beta|\Lambda_N|\left\langle(q_{12}-q)^2\right\rangle_{t,\lambda_0-t}\right\}\\
&\overset{\text{int.by_parts}}{=}\frac{\beta}{4|\Lambda_N|}\mathbb{E}\left\{\frac{\beta}{|\Lambda_N|}\sum_{i,j\in\Lambda_N}\left(1+\left\langle\sigma_i^1\sigma_j^1\sigma_i^2\sigma_j^2\right\rangle_{t,\lambda_0-t}\right.\right.\\
&\qquad\left.-2\left\langle\sigma_i^1\sigma_j^1\sigma_i^3\sigma_j^3\right\rangle_{t,\lambda_0-t}^{\otimes 2}\right)-2\beta q\sum_{i\in\Lambda_N}\left(1+\left\langle\sigma_i^1\sigma_i^2\right\rangle_{t,\lambda_0-t}\right.\\
&\qquad\left.\left.-2\left\langle\sigma_i^1\sigma_i^3\right\rangle_{t,\lambda_0-t}^{\otimes 2}\right)-\beta|\Lambda_N|\left\langle(q_{12}-q)^2\right\rangle_{t,\lambda_0-t}\right\}\\
&=\frac{\beta^2}{4}(1-q)^2-\frac{\beta^2}{2}\mathbb{E}\left\langle(q_{13}-q)^2\right\rangle_{t,\lambda_0-t}^{\otimes 2},
\end{aligned}$$

so that, integrating,

$$p_N^{(2)}(t,\lambda)\leqslant\frac{\beta^2}{4}(1-q)^2t+\frac{1}{2|\Lambda_N|}\mathbb{E}\log Z^{(2)}(0,t+\lambda). \tag{14}$$

Now, setting $u_N(t)=p_N^{\mathrm{RFIM}}(\kappa,h,\beta\sqrt{q})+(\beta^2/4)(1-q)^2t-p_N(t)$, which is a non-negative function by (12), and using convexity of the pressure with respect to λ and the identity $p_N^{(2)}(t,0)=p_N(t)$, we obtain for any $\lambda>0$

$$\begin{aligned}\frac{d}{dt}u_N(t) &\overset{(12)}{=} \frac{\partial}{\partial\lambda}p_N^{(2)}(t,0)\\ &\overset{\text{convexity}}{\leqslant} \frac{p_N^{(2)}(t,\lambda)-p_N(t)}{\lambda}\\ &\overset{(14)}{\leqslant} \frac{1}{\lambda}\left[u_N(t)+\frac{1}{2|\Lambda_N|}\mathbb{E}\log\frac{Z^{(2)}(0,t+\lambda)}{Z^{(2)}(0,0)}\right].\end{aligned} \tag{15}$$

Since $Z^{(2)}$ is increasing in λ, Eq. (15) implies

$$\frac{d}{dt}\log\left[u_N(t)+\frac{1}{2|\Lambda_N|}\mathbb{E}\log\frac{Z^{(2)}(0,1+\lambda)}{Z^{(2)}(0,0)}\right]\leqslant\frac{1}{\lambda},$$

which, recalling that $u_N(0)=0$, can be immediately integrated to give

$$u_N(t)\leqslant\left(e^{t/\lambda}-1\right)\times\frac{1}{2|\Lambda_N|}\mathbb{E}\log\frac{Z^{(2)}(0,1+\lambda)}{Z^{(2)}(0,0)} \tag{16}$$

and Eq. (5) follows if we can prove that

$$\lim_{N\to\infty}\frac{1}{2|\Lambda_N|}\mathbb{E}\log\frac{Z^{(2)}(0,\lambda_0)}{Z^{(2)}(0,0)}=0 \tag{17}$$

for some $\lambda_0>1$, if q is chosen properly. (Note that $\lambda_0=\lambda+1>1$ is required so that one can take t up to 1 and still have $\lambda>0$, which is needed in (15).)

Define for any $\mu\in\mathbb{R}$, $q\geqslant 0$

$$\alpha_N(\mu;J)=\frac{1}{2|\Lambda_N|}\log\left\langle e^{\mu|\Lambda_N|(q_{12}-q)}\right\rangle^{\otimes 2}_{\kappa,\beta\sqrt{q},N}$$

and $\alpha_N(\mu)=\mathbb{E}\,\alpha_N(\mu;J)$. We denote by $\langle.\rangle^{\otimes 2}_{\kappa,\beta\sqrt{q},N}$ the Gibbs measure for two replicas of the RFIM with parameters κ and $\gamma=\beta\sqrt{q}$ and volume Λ_N. Later we will also use the notation $\langle.\rangle^{(\mu)}_{\kappa,\beta\sqrt{q},N}$ for

$$\langle A\rangle^{(\mu)}_{\kappa,\beta\sqrt{q},N}=\frac{\left\langle A\,e^{\mu|\Lambda_N|(q_{12}-q)}\right\rangle^{\otimes 2}_{\kappa,\beta\sqrt{q},N}}{\left\langle e^{\mu|\Lambda_N|(q_{12}-q)}\right\rangle^{\otimes 2}_{\kappa,\beta\sqrt{q},N}}.$$

Let $\bar{q}_N = \bar{q}_N(\beta, \kappa, h)$ be the value, which minimizes

$$p_N^{\mathrm{RFIM}}(\kappa, h, \beta\sqrt{q}) + \frac{\beta^2}{4}(1-q)^2$$

with respect to q, cf (5). Clearly, $\bar{q}_N$ satisfies the "self-consistent equation"

$$q = \mathbb{E}\,\langle q_{12}\rangle^{\otimes 2}_{\kappa,\beta\sqrt{q},N} = \frac{\sum_{i\in\Lambda_N} \mathbb{E}\,\langle \sigma_i\rangle^2_{\kappa,\beta\sqrt{q},N}}{|\Lambda_N|}. \tag{18}$$

An analysis analogous to the one of Section 3.2 shows that the solution of (18) is unique for κ small enough, for N sufficiently large, and that $\bar{q}_N \to \bar{q}$ for $N \to \infty$. A Taylor expansion around $\mu = 0$ gives immediately

$$\alpha_N(\mu; J) = \alpha_N(0; J) + \mu\alpha'_N(0; J) + \int_0^\mu dy \int_0^y du\, \alpha''_N(u; J),$$

where

$$\alpha_N(0; J) = 0, \tag{19}$$

$$\alpha'_N(0; J) = \frac{1}{2}\left(\langle q_{12}\rangle^{\otimes 2}_{\kappa,\beta\sqrt{\bar{q}_N},N} - \mathbb{E}\,\langle q_{12}\rangle^{\otimes 2}_{\kappa,\beta\sqrt{\bar{q}_N},N}\right), \tag{20}$$

$$\alpha''_N(u; J) = \frac{|\Lambda_N|}{2}\left(\left\langle q_{12}^2\right\rangle^{(u)}_{\kappa,\beta\sqrt{\bar{q}_N},N} - \left(\langle q_{12}\rangle^{(u)}_{\kappa,\beta\sqrt{\bar{q}_N},N}\right)^2\right) \tag{21}$$

$$= \frac{1}{2|\Lambda_N|}\sum_{i,j\in\Lambda_N}\left\langle\left(\sigma_i^1\sigma_i^2 - \left\langle\sigma_i^1\sigma_i^2\right\rangle\right)\left(\sigma_j^1\sigma_j^2 - \left\langle\sigma_j^1\sigma_j^2\right\rangle\right)\right\rangle^{(u)}_{\kappa,\beta\sqrt{\bar{q}_N},N}. \tag{22}$$

We can view $\langle . \rangle^{(u)}_{\kappa,\beta\sqrt{\bar{q}_N},N}$ as the Gibbs measure of a system with $2|\Lambda_N|$ spins, with exponentially decaying pair interactions. Since κ is small, taking μ itself sufficiently small, we keep this system inside the Dobrushin uniqueness region. Using exponential decay of correlations as in Section 3.1, we obtain that $\alpha''_N(u; J)$ is bounded above by a constant, uniformly in N,J and in $u \in [0, \mu]$, so that

$$\alpha_N(\mu; J) \leqslant \mu\alpha'_N(0; J) + C\mu^2 \tag{23}$$

for $\kappa < \kappa_0(h)$. Note that this bound holds for any μ, since it does for small μ and the function α can grow at most linearly at infinity.

With this in hand, we go back to proving (17). After a Gaussian transformation

$$\frac{Z^{(2)}(0,\lambda_0)}{Z^{(2)}(0,0)} = \int dz \sqrt{\frac{|\Lambda_N|}{2\pi}} e^{-|\Lambda_N|\frac{z^2}{2}} \left\langle e^{\beta z\sqrt{\lambda_0}|\Lambda_N|(q_{12}-\bar{q}_N)} \right\rangle^{\otimes 2}_{\kappa,\beta\sqrt{\bar{q}_N},N}$$
$$= \int dz \sqrt{\frac{|\Lambda_N|}{2\pi}} e^{|\Lambda_N|(-z^2/2+2\alpha_N(\beta z\sqrt{\lambda_0};J))}.$$

Equations (23) and (20) imply that, for $\beta < \beta_0(h) = 1/\sqrt{4C\lambda_0}$

$$\frac{1}{2|\Lambda_N|}\mathbb{E}\log\frac{Z^{(2)}(0,\lambda_0)}{Z^{(2)}(0,0)} \leqslant \frac{1}{2|\Lambda_N|}\mathbb{E}\log\int dz\sqrt{\frac{|\Lambda_N|}{2\pi}}$$
$$\times e^{|\Lambda_N|\left(-z^2/2(1-4C\beta^2\lambda_0)+2z\beta\sqrt{\lambda_0}\alpha'_N(0;J)\right)}$$
$$\leqslant C'\mathbb{E}\,(\alpha'_N(0;J))^2$$
$$= \frac{C'}{4|\Lambda_N|^2}\sum_{i,j\in\Lambda_N}\mathbb{E}((\langle\sigma_i\rangle^2 - \mathbb{E}\,\langle\sigma_i\rangle^2)(\langle\sigma_j\rangle^2$$
$$-\mathbb{E}\langle\sigma_j\rangle^2)), \tag{24}$$

where for simplicity we have written $\langle . \rangle$ for $\langle . \rangle^{\otimes 2}_{\kappa,\beta\sqrt{\bar{q}_N},N}$. We now show that the last expression is of order $1/|\Lambda_N|$. Indeed, take two distinct sites i, j and consider the d-dimensional ball B_{ij} of radius $|i-j|/2$ centered at i. If $\langle\sigma_i\rangle$ were depending only on $(J_k, k\in B_{ij})$ and $\langle\sigma_j\rangle$ on $(J_k, k\in B^c_{ij})$ only, the corresponding term in (24) would be zero by independence. But in the Dobrushin region, we can approximate $\langle\sigma_i\rangle$ by the expectation of σ_i for the finite volume RFIM on B_{ij} with an error which is exponentially small in the radius $|i-j|/2$, uniformly in the J_k's. Doing similarly with $\langle\sigma_j\rangle$, we conclude that

$$\mathbb{E}((\langle\sigma_i\rangle^2 - \mathbb{E}\,\langle\sigma_i\rangle^2)(\langle\sigma_j\rangle^2 - \mathbb{E}\langle\sigma_j\rangle^2)) \leqslant C'\rho^{-C''|i-j|}$$

for suitable constants $C', C'' > 0$. This, together with (24), immediately implies that

$$\frac{1}{2|\Lambda_N|}\mathbb{E}\log\frac{Z^{(2)}(0,\lambda_0)}{Z^{(2)}(0,0)} \leqslant \frac{c}{|\Lambda_N|}.$$

At this point, recalling Eq. (16), one finds

$$p_N(\kappa,\beta,h) \geqslant p_N^{RFIM}(\kappa,h,\beta\sqrt{\bar{q}_N}) + \frac{\beta^2}{4}(1-\bar{q}_N)^2 - \left(e^{1/(\lambda_0-1)}-1\right)\frac{c}{|\Lambda_N|},$$

which, together with (13), proves the convergence in average of the pressure to the expression (5). Moreover, from Eqs. (15) and (12) one deduces that

$$\limsup_{N\to\infty} \sup_{0\leqslant t\leqslant 1} |\Lambda_N| \mathbb{E} \left\langle (q_{12}-\bar{q}_N)^2 \right\rangle_t^{\otimes 2} < \infty \,, \tag{25}$$

which will be needed in Section 3.5, where we deal with pressure fluctuations.

3.5. Proof of Theorem 3

We follow the strategy which was introduced in ref. 11, adapted to the present case where short-range interactions are also present. Let

$$\hat{f}_N(t) = \sqrt{|\Lambda_N|} \left(\frac{\log Z(t)}{|\Lambda_N|} - p_N(t) \right),$$

where $Z(t), p_N(t)$ were defined in Eqs. (11) and (12) and it is understood that q is taken to be $\bar{q}_N = \bar{q}_N(\beta, \kappa, h)$ as in Eq. (18), to be distinguished from $\bar{q} = \lim \bar{q}_N$.

We will prove that

$$\lim_{N\to\infty} \mathbb{E}\, e^{iu\hat{f}_N(t)} = \exp\left(-\frac{u^2}{2}(\Gamma - \frac{\beta^2}{2} t\bar{q}^2) \right) \tag{26}$$

for any $u \in \mathbb{R}$, $0 \leqslant t \leqslant 1$, from which Eq. (6) follows for $t=1$.

By means of integrations by parts one finds

$$\begin{aligned}
\partial_t \mathbb{E}\, e^{iu\hat{f}_N(t)} &= iu\mathbb{E}\, e^{iu\hat{f}_N(t)} \frac{d}{dt} \hat{f}_N(t) \\
&= \frac{\beta^2}{4} u^2 \bar{q}_N^2 \mathbb{E}\, e^{iu\hat{f}_N(t)} - \frac{\beta^2}{4} u^2 \mathbb{E}\, e^{iu\hat{f}_N(t)} \left\langle (q_{12}-\bar{q}_N)^2 \right\rangle_t^{\otimes 2} \\
&\quad - i\frac{\beta^2}{4} u \sqrt{|\Lambda_N|} \mathbb{E}\, e^{iu\hat{f}_N(t)} \left(\left\langle (q_{12}-\bar{q}_N)^2 \right\rangle_t^{\otimes 2} - \mathbb{E} \left\langle (q_{12}-\bar{q}_N)^2 \right\rangle_t^{\otimes 2} \right)
\end{aligned}$$

and, using Eq. (25),

$$\partial_t \mathbb{E}\, e^{iu\hat{f}_N(t)} = \frac{\beta^2 u^2 \bar{q}^2}{4} \mathbb{E}\, e^{iu\hat{f}_N(t)} + o(1).$$

Integrating in t, Eq. (26) is then proven provided that we show that

$$\lim_{N\to\infty} \mathbb{E}\, e^{iu\hat{f}_N(0)} = \exp\left(-\frac{u^2}{2}\Gamma\right), \tag{27}$$

i.e., a central limit theorem for pressure fluctuations of the RFIM at high temperature.

To this purpose, we employ the central limit theorem for martingales,[(13)] which we recall for convenience (similar ideas were employed in ref. 2, Section 6). Let $(\Omega, \mathcal{A}, P)$ be a probability space, $\mathcal{F}^{(n)} = \{\mathcal{F}_k^n\}_{0\leqslant k\leqslant n}$ a filtration of $\mathcal{A}$, for $n \in \mathbb{N}$, such that $\mathcal{F}_0^n = \{\emptyset, \Omega\}$, and $\xi^{(n)} = \{\xi_{n,k}\}_{1\leqslant k\leqslant n}$ a sequence of random variables adapted to $\mathcal{F}^{(n)}$. We denote by $E^{n,k}$ (respectively, $P^{n,k}$) the expectation (respectively, the probability) conditioned to $\mathcal{F}_k^n$ and by $V^{n,k}$ the conditional variance

$$V^{n,k}(X) = E^{n,k}(X^2) - \left(E^{n,k} X\right)^2$$

of a random variable X. We say that the triangular array $\{\xi_{n,k}\}_{n>0,1\leqslant k\leqslant n}$ is asymptotically negligible if for any $\varepsilon > 0$

$$\sum_{k=1}^{n} P^{n,k-1}\left(|\xi_{n,k}| \geqslant \varepsilon\right) \xrightarrow{\mathrm{P}} 0, \tag{28}$$

when $n \to \infty$. Then, the following holds:

Theorem 4. Let $\{\xi_{n,k}\}_{n>0,1\leqslant k\leqslant n}$ be an asymptotically negligible triangular array of square integrable random variables, and assume that for some $\Gamma > 0$,

$$\sum_{k=1}^{n} E^{n,k-1}(\xi_{n,k}) \xrightarrow{\mathrm{P}} 0 \tag{29}$$

and

$$\sum_{k=1}^{n} V^{n,k-1}(\xi_{n,k}) \xrightarrow{\mathrm{P}} \Gamma \tag{30}$$

for $n \to \infty$. Then,

$$\sum_{k=1}^{n} \xi_{n,k} \xrightarrow{\mathrm{law}} \mathcal{N}(0, \Gamma).$$

To simplify notations in our case, let $|\Lambda_N| = n$, $h + \beta\sqrt{\bar{q}_N} J_i = h_i$ and

$$p_{n,\underline{h}} = p_N^{\mathrm{RFIM}}(\kappa, h, \beta\sqrt{\bar{q}_N}; J).$$

Introducing the usual lexicographic ordering of the sites in Λ_N, we define $\mathcal{F}_k^n$, for $k \in \Lambda_N$, as the σ-algebra generated by the random fields J_i for $i \in \Lambda_N, i \preceq k$, and

$$\xi_{n,k} = \sqrt{n}(E^{n,k} p_{n,\underline{h}} - E^{n,k-1} p_{n,\underline{h}}),$$

so that

$$\sqrt{|\Lambda_N|}(p_N^{\mathrm{RFIM}}(\kappa, h, \beta\sqrt{\bar{q}_N}; J) - \mathbb{E}\, p_N^{\mathrm{RFIM}}(\kappa, h, \beta\sqrt{\bar{q}_N}; J)) = \sum_{k \in \Lambda_N} \xi_{n,k}. \tag{31}$$

Of course, $E^{n,k}(.) = \mathbb{E}_{J_\ell, \ell \in \Lambda_N, \ell \succ k}(.)$, and by convention $\mathcal{F}_k^n = \{\emptyset, \Omega\}$ if k precedes the first site in Λ_N. One can rewrite

$$\xi_{n,k} = -\frac{1}{\sqrt{n}} \mathbb{E}_{J'_k, J_\ell, \ell \succ k} \log \left\langle e^{(h'{}_k - h_k)\sigma_k} \right\rangle_{n,\underline{h}},$$

where J'_k is an independent copy of J_k – independent of $(J_l, l \in \mathbb{Z}^d)$ – , and $h'_k = h + \beta\sqrt{\bar{q}_N} J'_k$, so that

$$\left|\xi_{n,k}\right| \leqslant n^{-1/2} \mathbb{E}_{J'_k} |h'_k - h_k| \leqslant C n^{-1/2} (|h_k| + C)$$

for some constant C and

$$E^{n,k-1} \left|\xi_{n,k}\right|^3 \leqslant C' n^{-3/2}.$$

This implies asymptotic negligibility (28), since

$$\sum_{k \in \Lambda_N} P^{n,k-1}\left(|\xi_{n,k}| \geqslant \varepsilon\right) \leqslant \frac{1}{\varepsilon^3} \sum_{k \in \Lambda_N} E^{n,k-1}\left(|\xi_{n,k}|^3\right) \leqslant \frac{C'}{\varepsilon^3 \sqrt{n}}.$$

In order to apply Theorem 4, we have to check conditions (29) and (30). The first one is evident, since

$$E^{n,k-1} \xi_{n,k} = 0,$$

identically. As for the second, notice that

$$V^{n,k-1}(\xi_{n,k}) = \frac{1}{n}\mathbb{E}_{J_k}\left(\mathbb{E}_{J'_k, J_\ell, \ell \succ k} \log\left\langle e^{(h'_k - h_k)\sigma_k}\right\rangle_{n,\underline{h}}\right)^2. \tag{32}$$

Let k correspond to a site "in the bulk" of Λ_N, i.e., assume that the distance between k and the boundary of Λ_N is larger than, say, $n^{1/(2d)}$. In this case, we will write $k \in B_N$. We want to replace $\langle . \rangle_{n,\underline{h}}$ in (32) with the unique infinite-volume Gibbs measure $\langle . \rangle_{\infty,\underline{h}}$. To this purpose, note preliminarily that

$$\left| \log \frac{\left\langle e^{(h'_k - h_k)\sigma_k}\right\rangle_{n,\underline{h}}}{\left\langle e^{(h'_k - h_k)\sigma_k}\right\rangle_{\infty,\underline{h}}} \right| \leqslant 2|h'_k - h_k|.$$

Moreover, thanks to Dobrushin's theorem,

$$\frac{\left\langle e^{(h'_k - h_k)\sigma_k}\right\rangle_{n,\underline{h}}}{\left\langle e^{(h'_k - h_k)\sigma_k}\right\rangle_{\infty,\underline{h}}} = 1 + \frac{e^{(h'_k - h_k)} - e^{-(h'_k - h_k)}}{\left\langle e^{(h'_k - h_k)\sigma_k}\right\rangle_{\infty,\underline{h}}}\delta_{n,\underline{h}} \tag{33}$$

with some $\delta_{n,\underline{h}}$ such that

$$\lim_{n\to\infty} \sup_{k \in B_N} \sup_{\underline{h}} |\delta_{n,\underline{h}}| \equiv \lim_{n\to\infty} \epsilon_n = 0.$$

Denoting by $A_{n,k}$ the event

$$A_{n,k} = \left\{ |h'_k - h_k| \leqslant \frac{1}{2}\log\left(\frac{1}{2\epsilon_n}\right)\right\}$$

and using the fact that $|\log(1+x)| \leqslant D|x|$ for $|x| \leqslant 1/2$ for some finite constant D, one can write

$$\left| \log \frac{\left\langle e^{(h'_k - h_k)\sigma_k}\right\rangle_{n,\underline{h}}}{\left\langle e^{(h'_k - h_k)\sigma_k}\right\rangle_{\infty,\underline{h}}} \right| \leqslant D\epsilon_n e^{2|h'_k - h_k|} + 2|h'_k - h_k| 1_{A^C_{n,k}}.$$

Therefore one has

$$V^{n,k-1}(\xi_{n,k}) = \frac{1}{n}\mathbb{E}_{J_k}\left(\mathbb{E}_{J'_k,J_\ell,\ell\succ k}\log\left\langle e^{(h'_k-h_k)\sigma_k}\right\rangle_{\infty,\underline{h}}\right)^2+\frac{1}{n}o(1)$$
$$= \frac{1}{n}\left(\phi(\theta_{-k}\underline{h})+o(1)\right),$$

where $o(1)\to 0$ for $n\to\infty$ uniformly in $k\in B_N$, θ_k is the shift of vector k and

$$\phi(\underline{h})=\mathbb{E}_{J_0}\left(\mathbb{E}_{J'_0,J_\ell,\ell\succ 0}\log\left\langle e^{(h'_0-h_0)\sigma_0}\right\rangle_{\infty,\underline{h}}\right)^2$$

the subscript $_0$ referring of course to the origin of the lattice $\mathbb{Z}^d$. Note that there is a residual n-dependence in ϕ, since the fields h_i are defined through $\bar{q}_N$, but this dependence is easily seen to be harmless thanks to the exponential decay of correlations inside Dobrushin's uniqueness region, and to the fact that $\bar{q}_N\to\bar{q}$. Finally, defining $\tilde{h}_i=h+\beta\sqrt{\bar{q}}J_i$, the ergodic theorem implies that, for almost every J,

$$\lim_{n\to\infty}\sum_{k\in\Lambda_N}V^{n,k-1}(\xi_{n,k})=\lim_{n\to\infty}\frac{1}{n}\sum_{k\in\Lambda_N}\phi(\theta_{-k}\underline{h})=\mathbb{E}\,\phi(\underline{\tilde{h}})\equiv\Gamma(\kappa,\beta,h),$$

where we used the fact that the contribution of the spins $k\notin B_N$ vanishes for $|\Lambda_N|\to\infty$. At this point, all the conditions necessary to apply Theorem 4 are fulfilled and, recalling (31), one has (27), which concludes the proof of Theorem 3.

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